Discrete Mathematics

Unit 6: Counting, Binomials and Discrete Probability

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Discrete Mathematics Lecture Notes

Acknowledgements

- These lecture notes contain some material from the following sources:
  - [Rosen]: *Discrete Mathematics and Its Applications* by K. Rosen, 5th Edition, Tata McGraw-Hill Edition.

Pigeonhole Principle

The Pigeonhole Principle: If $k+1$ or more objects are placed into k boxes there is at least one box containing two or more of the objects.

The Generalised Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least N/k objects.

Permutations

A **permutation** of a set of distinct objects is an ordered arrangement of those objects. An ordered arrangement of r elements of a set is called an ***r*-permutation**.

The number of r-permutations of a set with n distinct elements is:

$$P(n, r) = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n - r)!}$$

The number of r-permutations of a set of n objects with repetition allowed is n^r.

The number of different permutations of n objects where there are n_1 distinguishable objects of type 1, n_2 distiguishable objects of type 2, …, and n_k distinguishable objects of type k, is:

$$\frac{n!}{n_1!\, n_2! \ldots n_k!}$$

Combinations

A **combination** of a set of distinct objects is an unordered selection of those objects. An unordered selection of *r* elements of a set is called an ***r*-combination**.

The number of *r*-combintaions of a set with *n* distinct elements is:

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

The number of *r*-combinations from a set with n elements when repetition of elements is allowed is $C(n+r-1, r)$ i.e.:

$$\frac{(n+r-1)!}{r!\,(n-1)!}$$

The Binomial Theorem

The equation $(x+y)^n$ can be expanded out simply multiplying $(x+y)$ n times.

The Binomial Theorem: If x and y are variables and n is a nonnegative integer then

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

Please note that $C(n, r)$ is often written as $^{n}C_{r}$ or as $\binom{n}{r}$

Discrete Probability

Let S be the sample space of an experiment with a finite or countable number of outcomes. We assign probability $p(x)$ to each outcome x. With n possible outcomes x_1, x_2, … and x_n, we have:

i) $0 \leq p(x_i) \leq 1$ for $i = 1, 2, \ldots, n$

ii) $\sum_{i=1}^{n} p(x_i) = 1$

The probability of the event E is the sum of the probabilities of the outcomes in E. $p(E) = \sum_{x \in E} p(x)$

$P(\overline{E}) = 1 - P(E)$

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$P(E \cap F) = P(E) \times P(F)$ given that E and F are independent

$P(E \cap F) = 0$ given that E and F are mutually exclusive

$P(E \mid F) = P(E \cap F) / P(F)$ conditional probability of E given F

Laplace's definition: $p(E) = \frac{m}{n} = \frac{|E|}{|S|}$ where m is the number of outcomes of E, out of n total outcomes

Bernoulli Trials and Binomial Distributions

A **Bernoulli trial** has two possible outcomes, a success or a failure. If p is the probability of success and q is the probability of failure then $p + q = 1$. E.g. with an unbiased coin the probability of heads (success) is 0.5 and that of tails (failure) is 0.5.

The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 – p$ is:

$$C(n, k)\ p^{k}\ q^{n-k}$$

For example, if you flip a coin 5 times, the probability of getting 3 heads (in any order) is $C(5, 3)\ 0.5^3\ 0.5^2 = 10 \times 0.5^5 = 0.3125$