Discrete Mathematics

Unit 3: Sets, Relations and Functions

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Discrete Mathematics Lecture Notes

Acknowledgements

- These lecture notes contain some material from the following sources:
  - [ICM]: *Introduction to Computer Mathematics* by C. Runciman, 2003
  - [Rosen]: *Discrete Mathematics and Its Applications* by K. Rosen, 5th Edition, Tata McGraw-Hill Edition.

Sets and Specifications

A **set** is a collection of distinct members (or elements).

A **set specification** is an expression of the form {member | predicate}. Or elements can simply be listed in any order between {…}. E.g. $\{p \mid \text{prime}(p) \wedge p<10\} = \{2, 3, 5, 7\}$.

N is the set of natural (non-negative whole) numbers i.e. {0, 1, 2, ….} whereas **Z** is the set of all whole numbers or integers {…. -2, -1, 0, 1, 2, …}.

The **empty set Ø** contains no elements that is $\emptyset = \{x \mid F\}$, whereas the **universal set U** contains all possible elements (in a domain or agreed world) that is $\mathbf{U} = \{x \mid T\}$

Set Membership and Subsets

Membership

$x \in S$ means "x is a member of S"

$x \notin S$ means "x is not a member of S" i.e. $\neg(x \in S)$

Axioms of membership:

$\{x \mid x \in S\} = S$ and $y \in \{x \,/\, p(x)\} \equiv p(y)$

Subset $\subseteq$: $P \subseteq Q$ is true if and only if all members of P are also members of Q. Remember, $\varnothing \subseteq P$ and $P \subseteq U$ for any set P.

Proper Subset $\subset$: $P \subset Q$ is true if $P \subseteq Q$ and $P \neq Q$

$$\begin{aligned} P \subseteq Q &\equiv \forall x, x \in P \Rightarrow x \in Q \\ P = Q &\equiv \forall x, x \in P \Leftrightarrow x \in Q \end{aligned}$$

Intersection, union, complement and disjoint sets

intersection $\cap$; union $\cup$

If P, Q are sets, their

$$\begin{aligned} \text{union} \quad P \cup Q &= \{x | x \in P \vee x \in Q\} \\ \text{intersection} \quad P \cap Q &= \{x | x \in P \wedge x \in Q\} \end{aligned}$$

difference $\setminus$

If P, Q are sets, their

$$\text{difference} \quad P \setminus Q = \{x | x \in P \wedge x \notin Q\}$$

complement

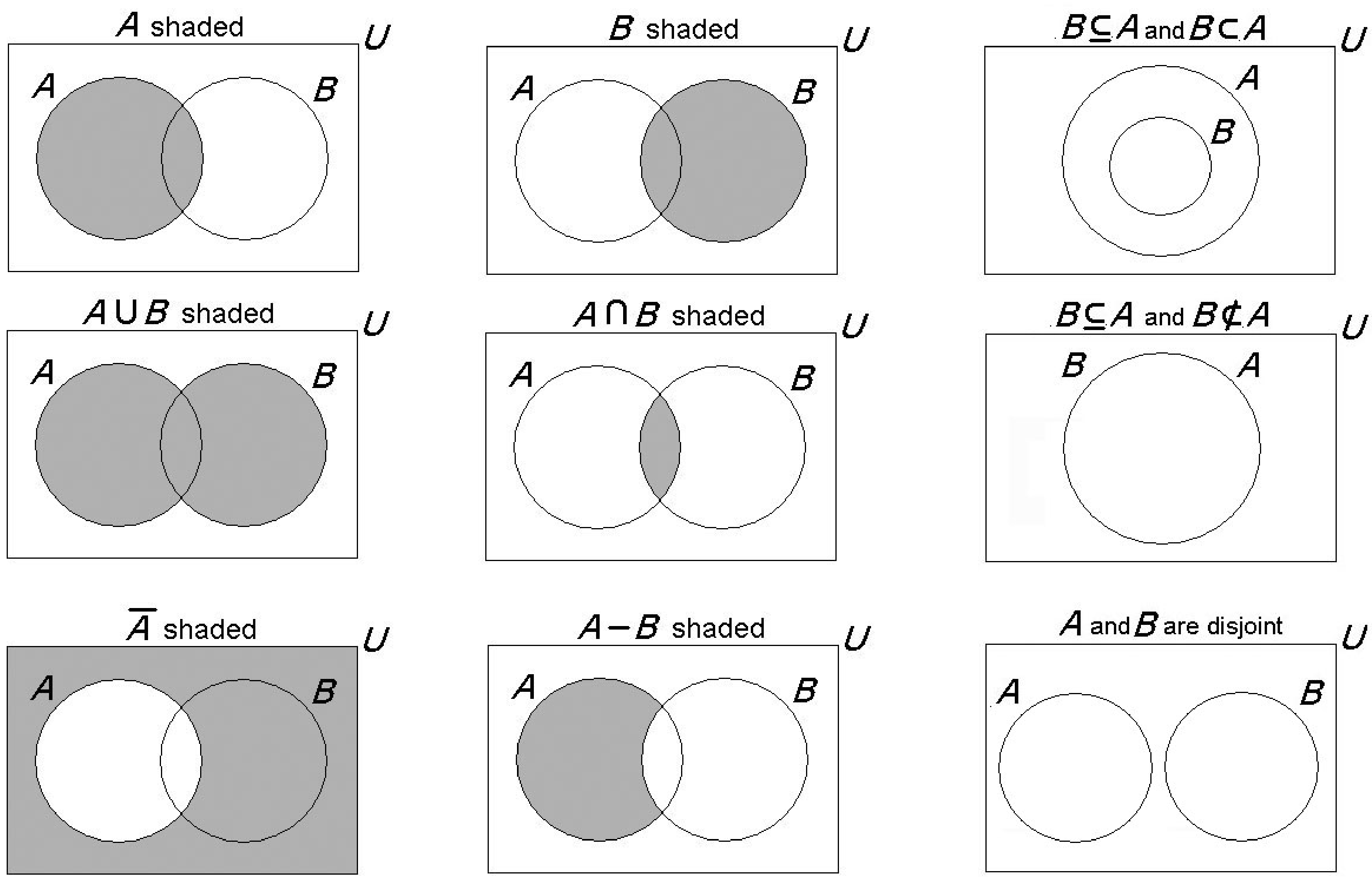
$$\text{complement} \quad \overline{P} = \{x | x \notin P\}$$

From [ICM]

Please note that difference $P \setminus Q$ is often written as $P - Q$, and the complement of P is also often written as P'.

Two sets P and Q are said to be **disjoint** if P and Q have no elements in common i.e. $P \cap Q = \varnothing$

Venn Diagrams



Theorems regarding sets

TABLE 1 Set Identities.

Identity	*Name*
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

From [Rosen]

Rules regarding set membership

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

From [Rosen]

Set Cardinality and Rank

If S is a set, $\#S$ (or card(S) or $|S|$) expresses its **cardinality** or the number of elements it has. E.g. $\#\emptyset = 0$ whereas $\#\{14, 6, 3, 2\} = 4$

Theorem *If P and Q are finite sets:*

(a) $Q \subseteq \bar{P} \Rightarrow \#(P \cup Q) = \#P + \#Q$

(b) $Q \subseteq P \Rightarrow \#(P \setminus Q) = \#P - \#Q$

Definition ($\uplus$, disjoint union) *$P \uplus Q$ is the union of disjoint sets P and Q.*

So, $\#(P \uplus Q) = \#P + \#Q$.

Theorem (union decomposition)

$P \cup Q = (P \setminus Q) \uplus (P \cap Q) \uplus (Q \setminus P)$

Theorem (cardinality rule)

$\#(P \cup Q) + \#(P \cap Q) = \#P + \#Q$.

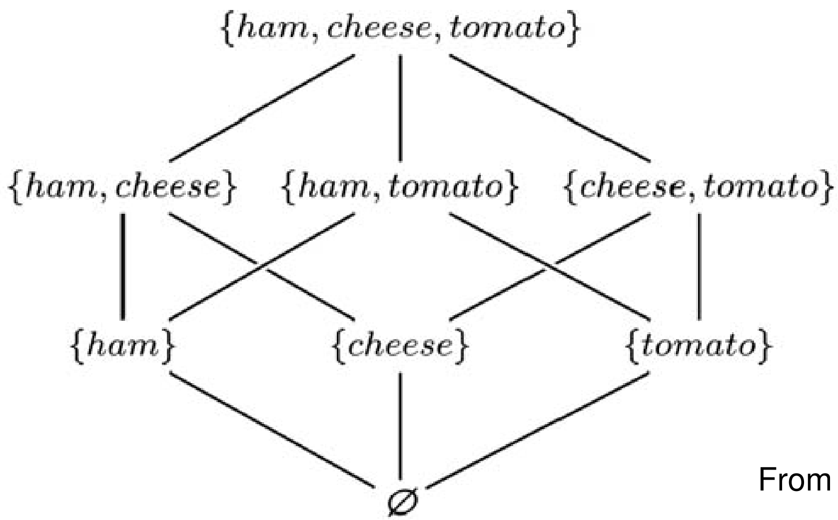
From [ICM]

Sets can contains other sets as elements e.g. $P = \{\{1,2\}, 3\}$. Anything other than a set (e.g. 3) has rank 0. If S is a set whose highest ranked element has rank k, then S has **rank** k+1. Therefore {1,2} has rank 0+1=1 and P has rank 1+1=2.

The Powerset

If S is a set, its **powerset** 2^S is the set which contains all possible subsets of S as its elements. i.e. $2^S = \{R \mid R \subseteq S\}$

E.g. S = {ham, cheese, tomato}, then 2^S includes:



From [ICM]

2^S = { Ø, {ham}, {cheese}, {tomato}, {ham, cheese}, {ham, tomato}, {cheese, tomato}, {ham, cheese, tomato} }

For any set S, its powerset 2^S will have $2^{\#S}$ elements. i.e. $\#(2^S) = 2^{\#S}$

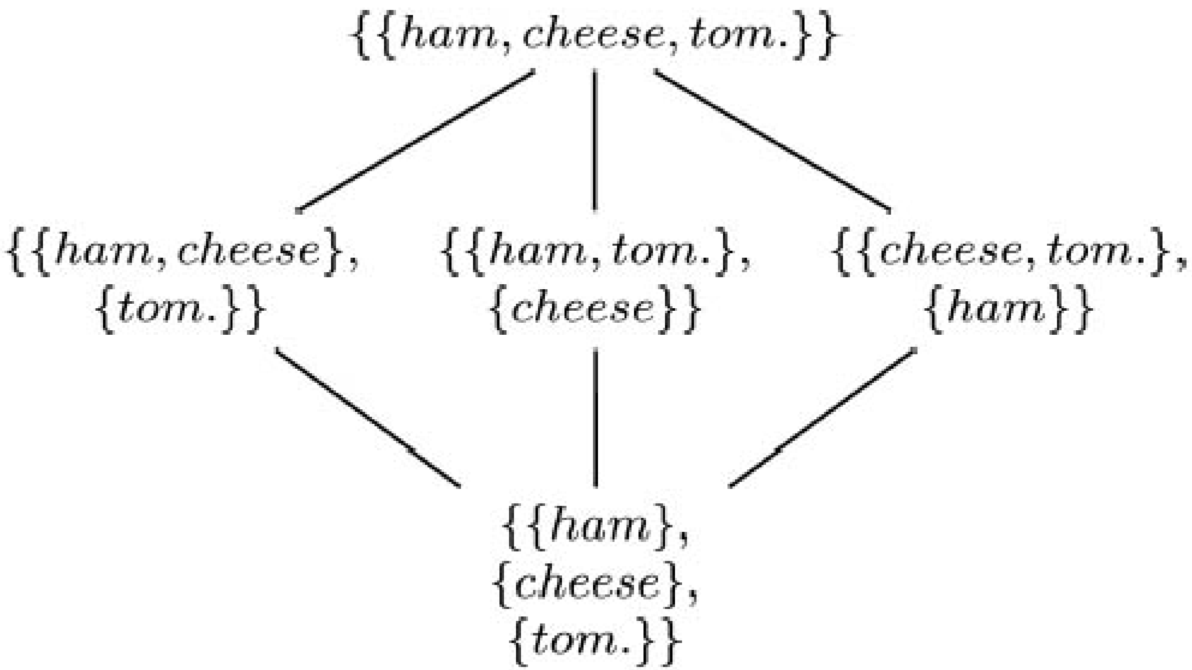
Partitions

A set $\{P_1, P_2, P_3, \ldots\}$ *of non-empty subsets of a set* S *is a* partition *of* S *exactly if:*

(1) $P_1 \cup P_2 \cup P_3 \cup \ldots = S$ *(*P_i*'s cover* S*), and*
(2) $j \neq k \Rightarrow P_j \cap P_k = \varnothing$ *(*P_i*'s are disjoint).*

The only partition of $\varnothing$ is itself $\varnothing$. If P is a partition of $S \neq \varnothing$ then $1 \leq \#P \leq \#S$.

the possible partitions of S *are*



From [ICM]

Cartesian Product and Relations

The **cartesian product** A x B of two sets A and B, is the set of ordered pairs $\{(a, b) \mid a \in A \wedge b \in B\}$.

E.g. D = {Mon, Tues, Wed, Thurs, Fri} and T = {am, pm}
D x T = { (Mon, am), (Mon, pm), (Tues, am), (Tues, pm), (Wed, am),
(Wed, pm), (Thurs, am), (Thurs, pm), (Fri, am), (Fri, pm) }

Therefore one can see that $\#(A \times B) = \#A \times \#B$

Given $A = \{0, 1\}$, B = {t, f} and C = {x, y}
A x B x C = {(0, t, x), (0, t, y), (0, f, x), (0, f, y), (1, t, x), (1, t, y), (1, f, x), (1, f, y) }
Therefore $\#(A \times B \times C) = \#A \times \#B \times \#C$

If $R \subseteq (A \times B)$, then R is called a **relation** from the set A to the set B.

Sets and Quantifiers

$\forall x \in S\ P(x)$ denotes the universal quantification of $P(x)$ where the universe of discourse is the set S. i.e.

$$\forall x \in S\ P(x) \equiv \forall x\ \ x \in S \Rightarrow P(x)$$

$\exists x \in S\ P(x)$ denotes the existential quantification of $P(x)$ where the universe of discourse is the set S.

$$\exists x \in S\ P(x) \equiv \exists x\ \ x \in S \wedge P(x)$$

Examples:

$\forall x \in R\ (x^2 \geq 0)$ means that the square of all real numbers is greater than or equal to 0.

$\exists x \in Z\ (x^2 = 4)$ means that there is at least one integer whose square is 4

Binary Relations

If $R \subseteq (A \times B)$, then R is called a **relation** from the set A to the set B i.e. $R : A \leftrightarrow B$. A relation may be represented using set specification, a directed graph or an adjacency matrix.

The **directed graph** for R contains nodes representing each element of A and B where an edge from x to y exists if $x \in A \wedge y \in B \wedge (x,y) \in R$ (which can also be written as xRy). The **adjacency matrix** M for R is a matrix of dimensions $\#A \times \#B$, where $M_{x,y} = 1$ if $x \in A \wedge y \in B \wedge (x,y) \in R$ (i.e. xRy) and 0 otherwise.

Throughout this course we will be dealing mostly with relations involving two sets, however n-ary relations involving n sets often arise. For a relation $R : A_1 \times A_2 \times \ldots \times A_n$, $A_1, A_2, \ldots, A_n$ are called its domains and n is called its degree. Each element of R will be a tuple $(a_1, a_2, \ldots, a_n)$. This is analogous to the way records are stored in database systems.

An example of a relation

E.g. D = {Mon, Tues, Wed, Thurs, Fri} and T = {am, pm}

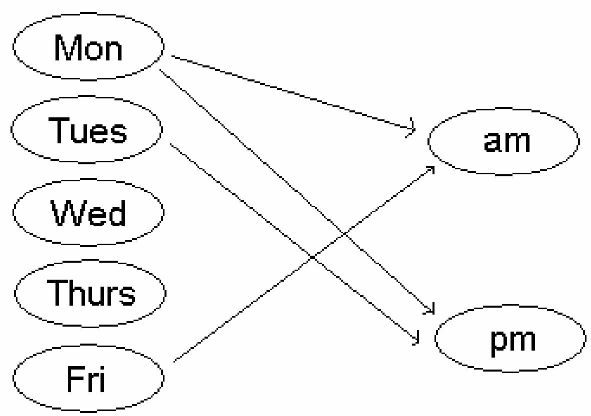
$D \times T$ = { (Mon, am), (Mon, pm), (Tues, am), (Tues, pm), (Wed, am), (Wed, pm), (Thurs, am), (Thurs, pm), (Fri, am), (Fri, pm) }

Specification of S where $S : D \leftrightarrow T$

S = { (Mon, am), (Mon, pm), (Tues, pm), (Fri, am) }

One can say, S(Mon) = {am, pm}, S(Fri) = {am}, S(Wed) = {}.

Diagraph G_S of S



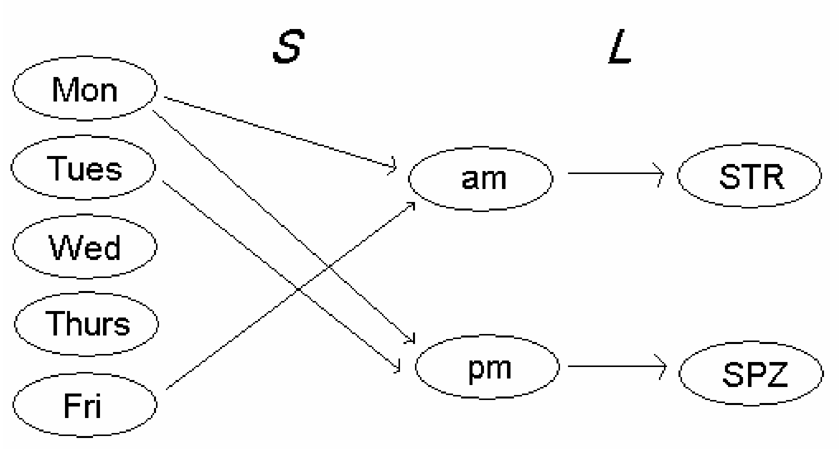
Adjacency matrix M_S for S

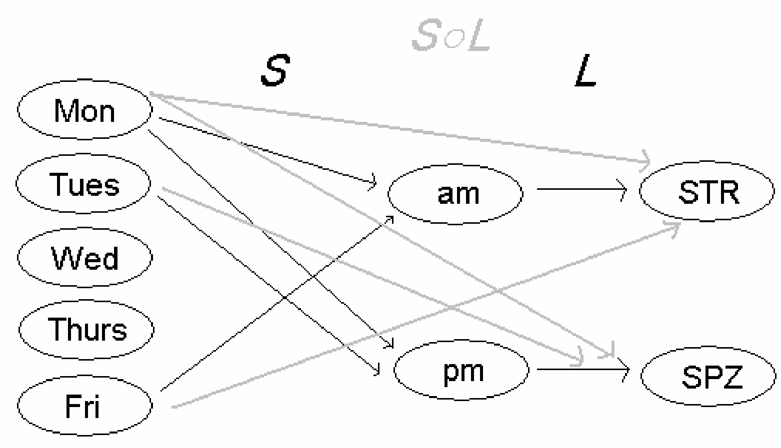
	am	pm
Mon	1	1
Tues	0	1
Wed	0	0
Thurs	0	0
Fri	1	0

Relational inverse and composition

If $R : A \leftrightarrow B$, then its relation inverse R^{-1} is defined by the rule $xRy \Leftrightarrow yR^{-1}x$. If M is the adjacency matrix of R, then M^T (i.e. the transpose of M) is the adjacency matrix for R^{-1}. The digraph for R^{-1} has all the edges from R only in the opposite direction. E.g. S^{-1}(am) = {Mon, Fri}, S^{-1}(pm) = {Mon, Tues}

For any $R : A \leftrightarrow B$ and S : $B \leftrightarrow C$, the **relational composition** $R \circ S : A \leftrightarrow C$ is defined by $(a,c) \in R \circ S \Leftrightarrow \exists b\ (a,b) \in R \wedge (b,c) \in S$. E.g. where $T = S \circ L$, T(Mon)= {STR, SPZ}, T(Thurs) = {}, T^{-1}(SPZ) = {Mon, Tues}.





Relational inverse and composition

Rules about relational inverse composition:

- Given a relation R, $M_{R^{-1}} = M_R^{\mathrm{T}}$
- Relations R and S can be combined using using basic set operations. For example:

  $(x,y) \in R \cap S \Leftrightarrow (x,y) \in R \wedge (x,y) \in R$, and $(x,y) \in R \cup S \Leftrightarrow (x,y) \in R \vee (x,y) \in R$
- If R and S are composable relations then $M_{R \circ S} = M_R . M_S$
- The adjacency matrix of $R \cup S$ can be computed as $M_R + M_S$
- During addition/multiplication of adjacency matrices, boolean rules are applied where sum(a,b) = max(a,b) and product(a,b) = min(a,b)
- Since, $(M_R . M_S)^{\mathrm{T}} = M_S^{\mathrm{T}} . M_R^{\mathrm{T}}$, $(R \circ S)^{-1} = (S^{-1} \circ R^{-1})$
- If R is a relation on the set A, i.e. $R : A \leftrightarrow A$, then

  $R^1 = R$, and $R^{n+1} = R^n \circ R$, i.e. $M_{R^{n+1}} = M_{R^n} . M_R$

Definitions of Relations

Consider a relation $R : A \leftrightarrow A$, with adjacency matrix M_R and digraph G_R

Reflexive/Irreflexive: Relation R is **reflexive** if $\forall x \in A$, $(x,x) \in R$, the diagonal of M_R contains all 1s and G_R contains loops around every node (e.g. $=$, $\leq$).

Relation R is **irreflexive** if $\forall x \in A$, $(x,x) \notin R$, the diagonal of M_R contains all 0s and G_R contains no loops around a node (e.g. $<$, $>$). Some relations are neither reflexive nor irreflexive.

Symmetric/Antisymmetric: Relation R is **symmetric** if $\forall x,y \in A$, $(x,y) \in R \Rightarrow (y,x) \in R$, the matrix M_R is symmetric along its diagonal and all edges in G_R are two-way (e.g. $=$, spouse).

Relation R is **anti-symmetric** if $\forall x,y \in A$, $(x,y) \in R \wedge (y,x) \in R \Rightarrow (x=y)$, any off-diagonal 1 in matrix M_R is mirrored by a 1 and no edges in G_R are two-way (e.g. parent, $<$). Some relations are neither symmetric nor anti-symmetric.

Definitions of Relations

Transitive: Relation R is **transitive**, $\forall x,y,z \in A\ (x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R$ and for every two-edge path between two nodes on G_R there is also a direct link between them (e.g. <, >, =). R is transitive iff $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Some other definitions:

The **identity relation** $I_A : A \leftrightarrow A$ is defined by $(x,y) \in I_A \Leftrightarrow (x=y)$. The digraph for I_A contains all loops around each node and its identity matrix contains 1s on its diagonal and 0 everywhere else. For any $R : A \leftrightarrow A$, $R^0 = I_A$

A **partial ordering** is a relation $R : A \leftrightarrow A$, that is reflexive, anti-symmetric and transitive (e.g. ≥, ≤). The pair (A, R) is called a **poset**. A strict partial ordering is irreflexive, anti-symmetric and transitive (e.g. <, >).

Elements x, y of poset (A, R) are said to be **comparable** if $xRy \vee yRx$. A **total/linear ordering** is one in which every pair of elements is comparable.

An **equivalence relation** is one that is reflexive, transitive and symmetric. E.g. =

If R on A is an equivalence relation, for each $x \in A$ we define the **equivalence class** $[x]_R = \{ y \mid xRy \}$

Closures of Relations

If $X \in$ {reflexive, symmetric, transitive}, the X closure of a relation $R : A \leftrightarrow A$, is the smallest relation on A with property X and R as a subset.

The **reflexive closure** of a relation R on A is $R \cup I_A$, or $R \cup R^0$

The **symmetric closure** of a relation R on A is $R \cup R^{-1}$

The **transitive closure** of a relation R on A is R^+. $R^+ = R \cup R^2 \cup \ldots \cup R^n$, ($n = \#A$)

One can also have closures with more than one property, for example:
The **reflexive transitive closure** of a relation R on A is R^*, where $R^* = R^0 \cup R^+$

Things to remember:

$R^+ \cup R^0$ reflexive closure of transitive closure	$=$	$(R \cup R^0)^+$ transitive closure of reflexive closure
$(R \cup R^0) \cup (R \cup R^0)^{-1}$ symmetric closure of reflexive closure	$=$	$(R \cup R^{-1}) \cup (R \cup R^{-1})^0$ reflexive closure of symmetric closure

BUT

$R^+ \cup (R^+)^{-1}$ symmetric closure of transitive closure	$\neq$	$(R \cup R^{-1})^+$ transitive closure of symmetric closure

The RHS is the symmetric transitive closure.

From [ICM]

Functions and their types

A relation $f : A \leftrightarrow B$, is a **function** if for every $x \in A$, there is at most one $y \in B$, such that $(x,y) \in f$. A function f is defined as $f : A \rightarrow B$, where A and B are called the **domain** and **codomain** respectively of f.

The **range**/**image** of f is the subset of B that is related by f to elements of A. The subset of A that is related by f to elements of B is called the **domain of definition** of f. If x is in the domain of definition of f, then there exists exactly one $y \in B$ such that $f(x)=y$. f is undefined for elements in A that are not in its domain of definition.

If f is a function, M_f will have at most one 1 in each row whereas G_f will have at most one edge coming out of each A node. A **partial function** ($f : A \rightarrow B$) may have some elements of A that are not related to elements in B, i.e. the domain and the domain of definition are not the same for f.

A function f is a **total function** ($f : A \rightarrow B$), if for every $x \in A$ there is exactly one $y \in B$ such that $(x,y) \in f$. M_f will have exactly one 1 in each row whereas G_f will have exactly one edge coming out of each A node. Usually when we speak of functions, we refer to total functions.

Functions and their types

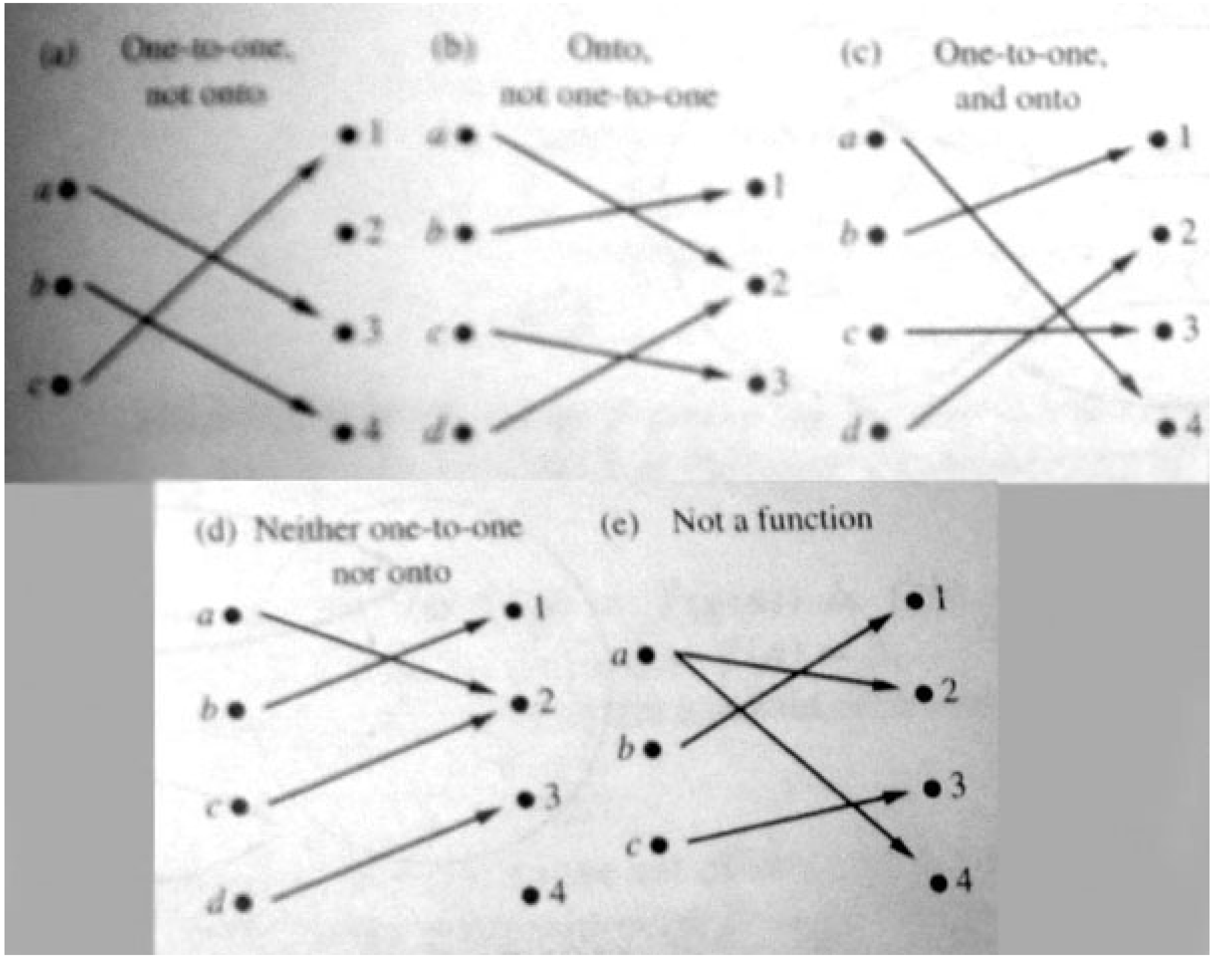
A function f is a **surjective/onto function** ($f : A \rightarrow B$), if for every $y \in B$ there is at least one $x \in A$, such that $(x,y) \in f$. M_f will have at least one 1 in each column whereas G_f will have at least one edge coming into each B node.

A function f is a **injective/one-to-one function** ($f : A \rightarrow B$), if for every $y \in B$ there is at most one $x \in A$ such that $(x,y) \in f$. M_f will have at most one 1 in each column whereas G_f will have at most one edge coming into each B node.

A function ($f : A \rightarrow B$) that is both surjective and injective is called a **bijection** or a **one-to-one correspondence**.

A **permutation** is a one-to-one correspondence on a finite set.

Examples of digraphs of functions



From [Rosen]