

# **Discrete Mathematics**

## ***Unit 2: Arguments, Inferences and Proofs***

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# *Discrete Mathematics Lecture Notes*

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## **Acknowledgements**

- These lecture notes contain some material from the following sources:
  - *Introduction to Computer Mathematics* by C. Runciman, 2003
  - *Discrete Mathematics and Its Applications* by K. Rosen, 5<sup>th</sup> Edition, Tata McGraw Hill Edition.

# *Arguments*

An **argument** is an assertion of the form  $P_1, P_2 \dots P_n \vdash Q$ , where propositions  $P_1, P_2 \dots, P_n$  are its premises and proposition  $Q$  is its conclusion. This can also be written as

$$\begin{array}{c} P_1 \\ P_2 \\ \vdots \\ \underline{P_n} \\ Q \end{array}$$

An argument is **valid** if whenever its premises are true, its conclusion is true; otherwise the argument is **fallacious**.

# *Valid Arguments?*

Consider the following argument:

Socrates is a man

All men are mortal

Socrates is mortal

Is this a valid argument?

# *Valid Arguments?*

Now, consider this argument:

Every foo is a bar

Every bar is a baz

Every foo is a baz

So, is this a valid argument?

# *Valid Arguments?*

Now, consider this argument:

At least one foo is a bar

At least one bar is a baz

At least one foo is a baz

So, is this a valid argument?

# *Valid Arguments?*

Would it be easier if you replaced foo, bar and baz with words that made sense e.g.

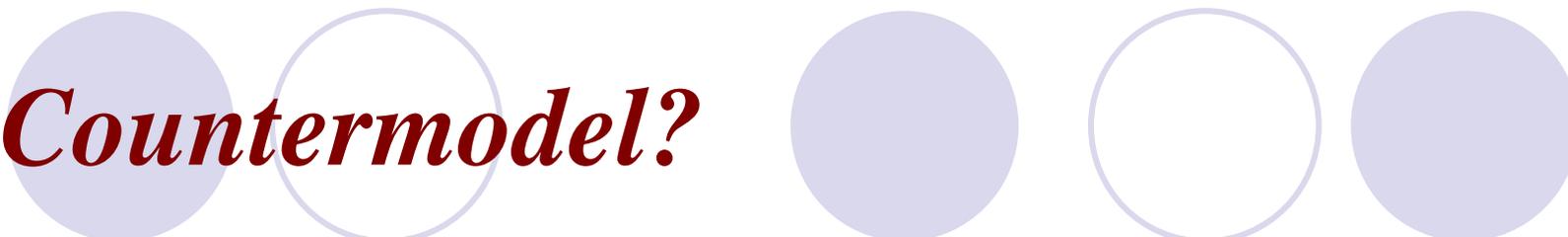
At least one student is a Teaching Assistant

At least one Teaching Assistant is a Lab Assistant

At least one student is a Lab Assistant

Does the argument seem valid now?

# *A Countermodel?*



The argument can also be proved not valid using a countermodel without any other knowledge of the domain.

Imagine a world that contains exactly two people:

1. a student who is a TA but not a Lab Assistant
2. a TA who is a Lab Assistant but not a student

Is the previous argument correct in this world?

# *Valid Arguments*

An argument  $P_1, P_2 \dots P_n \vdash Q$  is **valid** if and only if  $P_1 \wedge P_2 \wedge \dots \wedge P_n \Rightarrow Q$  is a tautology (i.e. always true). Otherwise they are called **fallacies**.

Argument 1:

If you read the book, you will pass the course

You read the book

---

You passed the course

Is this argument valid?

$p$  *You read the book*

$q$  *You passed the course*

# *Valid or Fallacious?*

$p$  You read the book

$q$  You passed the course

Argument 2:

If you read the book, you will pass the course

You did not read the book

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You did not pass the course

Argument 3:

If you read the book, you will pass the course

You passed the course

---

You read the book

# *Valid Arguments*

Consider the following arguments:

Modus ponens is the argument  $p, p \Rightarrow q \vdash q$ .

Modus tollens is the argument  $p \Rightarrow q, \neg q \vdash \neg p$ .

reductio ad absurdum is the argument  $p \Rightarrow q, p \Rightarrow \neg q \vdash \neg p$

Are they valid?

# *Rules of Inference*



Rules of inference can also be used to show how the conclusion of an argument can be derived from.

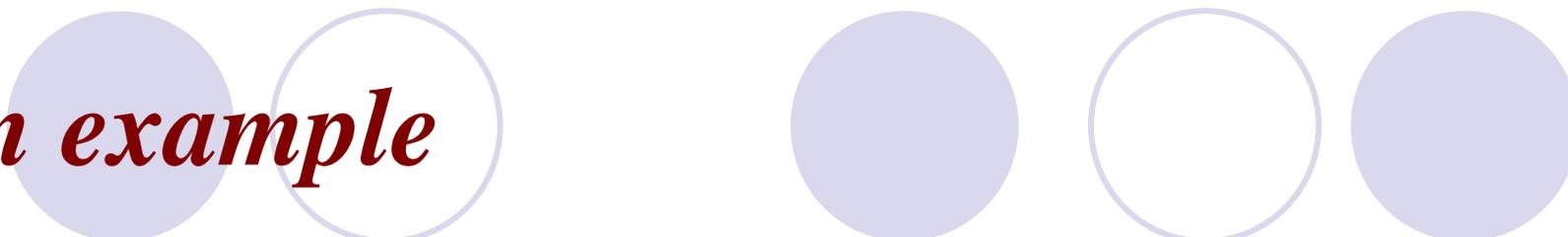
Using such rules one can show how the conclusion must be true if the premises are true, thereby proving that  $P_1 \wedge P_2 \wedge \dots \wedge P_n \Rightarrow Q$  is a tautology and the argument is valid

# Rules of Inference

Rosen pg57

**TABLE 1 Rules of Inference.**

<i>Rule of Inference</i>	<i>Tautology</i>	<i>Name</i>
$\frac{p}{\therefore p \vee q}$	$p \rightarrow (p \vee q)$	Addition
$\frac{p \wedge q}{\therefore p}$	$(p \wedge q) \rightarrow p$	Simplification
$\frac{p}{\therefore p \wedge q}$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$\frac{p \quad p \rightarrow q}{\therefore q}$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus ponens
$\frac{\neg q \quad p \rightarrow q}{\therefore \neg p}$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus tollens
$\frac{p \rightarrow q \quad q \rightarrow r}{\therefore p \rightarrow r}$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$\frac{p \vee q \quad \neg p}{\therefore q}$	$[(p \vee q) \wedge \neg p] \rightarrow q$	Disjunctive syllogism
$\frac{p \vee q \quad \neg p \vee r}{\therefore q \vee r}$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$	Resolution



## *An example*

If you email me, I will finish programming

If you do not email me, I will sleep early

If I sleep early, I will wake up refreshed

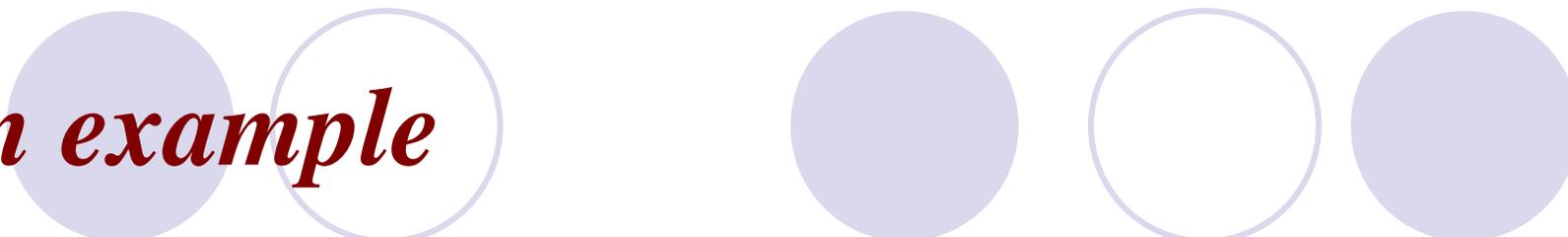
If I do not finish programming, I will wake up refreshed

$p \equiv$  You email me

$q \equiv$  I finish programming

$r \equiv$  I sleep early

$s \equiv$  I wake up refreshed



## *An example*

If you email me, I will finish programming

If you do not email me, I will sleep early

If I sleep early, I will wake up refreshed

---

If I finish programming, I will not wake up refreshed

$p$   $\equiv$  You email me

$q$   $\equiv$  I finish programming

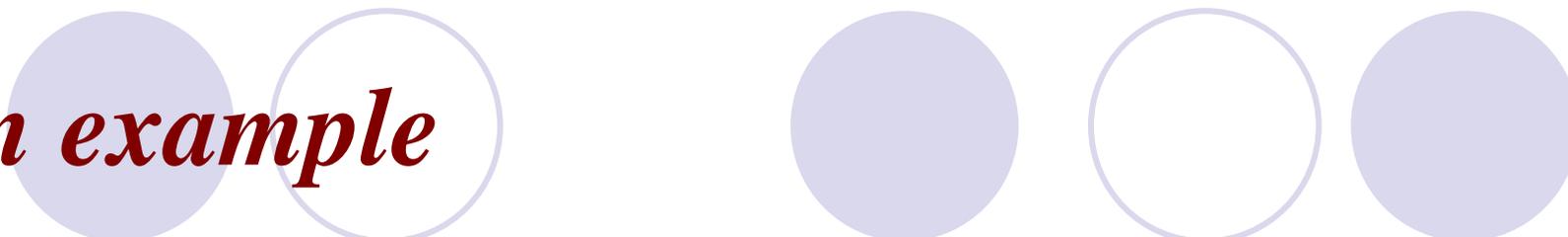
$r$   $\equiv$  I sleep early

$s$   $\equiv$  I wake up refreshed

# *Inference Rules with Quantified Statements*

Rosen pg59

<i>Rule of Inference</i>	<i>Name</i>
$\frac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization
$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$	Existential generalization



## *An example*

A student in this class has not read the book

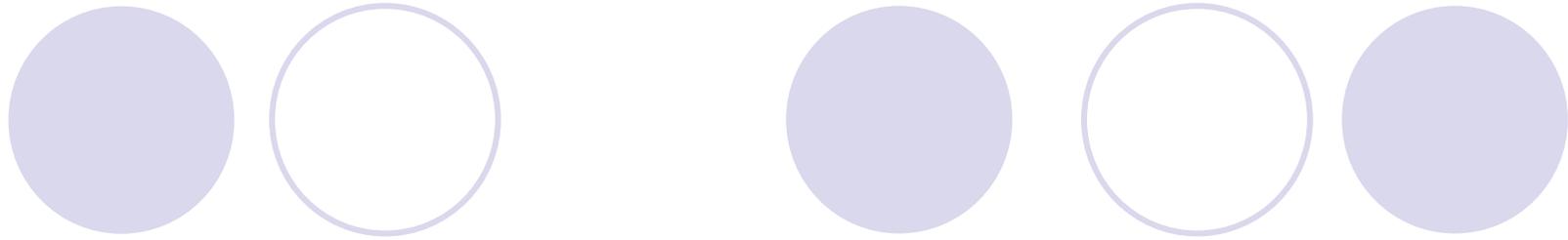
Everyone in this class passed the first exam

Someone who passed the first exam hasn't read the book

$C(x)$   $x$  is in this class

$B(x)$   $x$  has read the book

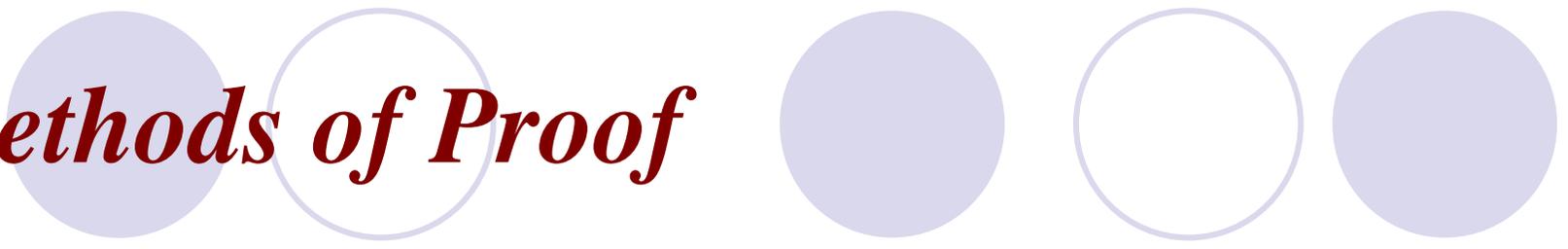
$P(x)$   $x$  has passed the first exam



Are the following arguments valid? Justify your answers using propositional logic.

- If I could find my house keys, I could get into the house. I can't find my keys. *Therefore* I can't get into the house.
- All horses are green. Billy is blue. *Therefore* Billy is not a horse.
- Study is necessary to understand logic. If I stay asleep, I don't study. But I shall not stay asleep. *Therefore* I shall understand logic.
- All academics are introverts. Unless I am an academic, I do not wear a tweed jacket. I am not an introvert. *Therefore* I do not wear a tweed jacket.

# *Methods of Proof*

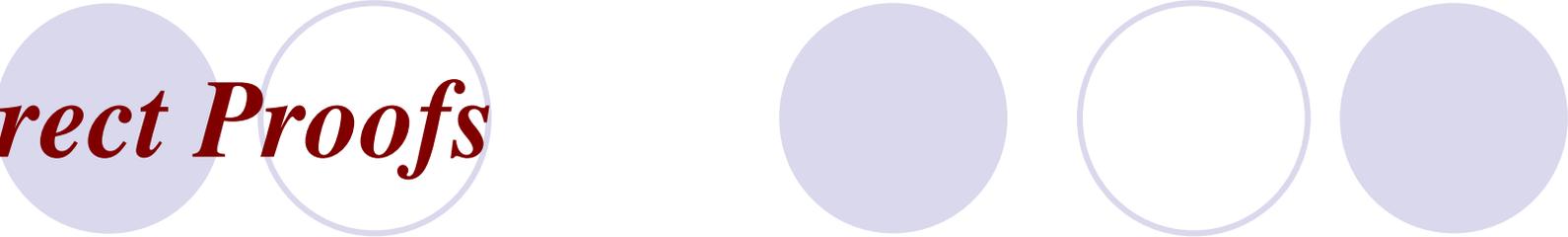
A decorative graphic consisting of six circles arranged in a horizontal line. The first circle is solid light blue. The second circle is white with a light blue outline. The third circle is solid light blue. The fourth circle is white with a light blue outline. The fifth circle is solid light blue. The sixth circle is solid light blue.

- Direct Proofs
- Indirect Proofs
- Vacuous Proofs
- Proof by Contradiction
- Proof by Cases
- Existence Proofs
- Counterexamples
- Uniqueness Proofs

# *Theorems and Proofs*

- A **theorem** is a statement that can be shown to be true
- One can demonstrate that a theorem is true with a sequence of statements that form an argument called a **proof**.
- The statements used in a proof include **axioms** or **postulates** (underlying assumptions about mathematical structures), while **rules of inference** tie together the steps of a proof.
- A **fallacy** is an argument that is not valid because of incorrect reasoning.
- A **lemma** is a simple theorem used in proving other theorems.
- A **corollary** is a proposition that can be established directly from a theorem that has been proved.
- A **conjecture** is a statement whose truth value is unknown. Once a conjecture is shown to be true via a proof it becomes a theorem
- For an implication  $p \Rightarrow q$ ,  $p$  is called the **hypothesis** and  $q$  is its **conclusion**.

# *Direct Proofs*



An implication  $p \Rightarrow q$ , can be proved by showing whenever  $p$  is true then  $q$  must be true.

For example, consider the following definitions:

A integer  $n$  is even if there exists another integer  $k$  such that  $n=2k$  and it is odd if there exists another integer  $k$  such that  $n=2k+1$ . A number can be either even or odd and not both.

Now we are told to attempt a direct proof of the theorem “If  $n$  is an odd integer then  $n^2$  is an odd integer.”

# *Indirect Proofs*

Since  $p \Rightarrow q$  is equivalent to its contrapositive  $\neg q \Rightarrow \neg p$ , one can prove it is true by proving that contrapositive is true.

Prove that “If  $3n+2$  is odd, then  $n$  is odd” via indirect proof



# *Vacuous and Trivial Proofs*

If for an implication  $p \Rightarrow q$ , its hypothesis  $p$  is always false then the statement  $p \Rightarrow q$  is always true.

E.g.  $P(x) \equiv x^2 > x$  where  $x$  is a natural number.

Prove that “ $P(0) \Rightarrow \forall n P(n)$ ”

# *Prove by Contradiction*



While proving  $p$  is true, suppose that a contradiction  $q$  can be found so that  $\neg p \Rightarrow q$  is true, that is  $\neg p \Rightarrow F$  is true. Then the proposition  $\neg p$  must be true. Consequently  $p$  is true. One can find a contradiction such that  $\neg p \Rightarrow r \wedge \neg r$ , therefore  $p$  is true.

Show that at least four of any 22 days must fall on the same day of the week.

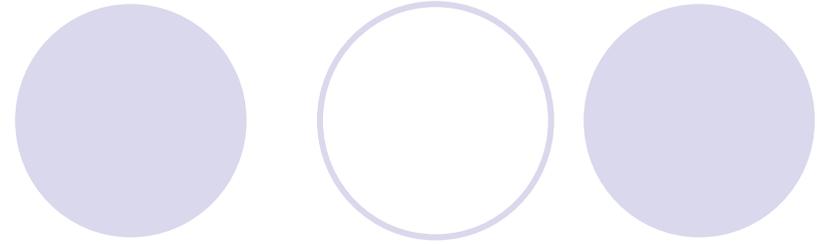
# *Prove by Contradiction*

While proving  $p$  is true, suppose that a contradiction  $q$  can be found so that  $\neg p \Rightarrow q$  is true, that is  $\neg p \Rightarrow F$  is true. Then the proposition  $\neg p$  must be true. Consequently  $p$  is true. One can find a contradiction such that  $\neg p \Rightarrow r \wedge \neg r$ , therefore  $p$  is true.

Show that at least four of any 22 days must fall on the same day of the week.

Proof: Let  $p$  be the proposition “at least four of any 22 days must fall on the same day of the week”. Suppose  $\neg p$  is true. i.e. at most three of any 22 days must fall on the same day of the week. Since a week has 7 days, this implies at most 21 days are picked. Therefore, we have a contradiction, making  $\neg p \Rightarrow r \wedge \neg r$  true, where  $r$  is the statement “22 days were chosen”. Therefore  $p$  is true.

# *Proof by Cases*



To prove an implication of the form:

$$(p_1 \vee p_2 \vee \dots p_n \Rightarrow q)$$

Since we know

$$[(p_1 \vee p_2 \vee \dots p_n \Rightarrow q)] \Leftrightarrow [(p_1 \Rightarrow q) \wedge (p_2 \Rightarrow q) \wedge \dots (p_n \Rightarrow q)]$$

All we have to do is prove all  $p_i \Rightarrow q$  are true where  $i = 1, 2, \dots, n$

For example: Use a proof by cases to show that  $|xy| = |x||y|$  where  $x$  and  $y$  are real numbers. Remember that  $|x|=x$ , if  $x \geq 0$  and  $|x|=-x$  if  $x < 0$



# *Prove of Equivalence*

In order to prove that  $p \Leftrightarrow q$ , one can use the tautology:

$$p \Leftrightarrow q \equiv p \Rightarrow q \wedge q \Rightarrow p$$

So by individually proving  $p \Rightarrow q$  is true and  $q \Rightarrow p$  is true, one can prove  $p \Leftrightarrow q$  is true as a whole.

# *Witnesses and Counter-examples*

## **Existential Proof**

For an existential formula,  $\exists x, p(x)$  a **witness** is a value of  $x$  making  $p(x)$  true, thereby proving  $\exists x, p(x)$  true as a whole.

## **Proof by Counter-example**

For a universal formula,  $\forall x, p(x)$  a **counter-example** is a value of  $x$  making  $p(x)$  false, thereby proving  $\forall x, p(x)$  false as a whole.

# *Uniqueness Proof*

Existence: We show that an element  $x$  with the desired property exists, i.e.  $\exists x p(x)$

Uniqueness: We show that if  $y \neq x$ , then  $y$  can not have the desired property, i.e.  $\exists x p(x) \wedge \forall y (x \neq y) \Rightarrow \neg p(y)$   
or  $\exists x p(x) \wedge \forall y p(y) \Rightarrow y=x$

Example: Show that every integer has an additive inverse, i.e. show that for every integer  $p$  there is a unique integer  $q$  such that  $p+q=0$

# *Uniqueness Proof*

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Example: Show that every integer has an additive inverse, i.e. show that for every integer  $p$  there is a unique integer  $q$  such that  $p+q=0$

Proof: If  $p$  is an integer, we find that  $p+q=0$  when  $q=-p$  and  $q$  is also an integer. Thus, we have proven the existence part.

To show that  $q$  is unique, suppose that  $r$  is an integer with  $r \neq q$  such that  $p+r=0$ . Then  $p+r=p+q$ . By subtracting  $p$  from both sides we have  $r=q$ , which contradicts our assumption  $r \neq q$ . Consequently, there is a unique integer  $q$  such that  $p+q=0$

# *Proof by Weak Natural Induction*

A proof by mathematical induction that  $P(n)$  is true for every positive integer  $n$  consists of two steps:

**BASIC STEP:** The proposition  $P(1)$  is shown to be true.

**INDUCTIVE STEP:** The implication  $P(k) \Rightarrow P(k+1)$  is shown to be true for every positive integer  $k$ .

$P(k)$  is called the inductive hypothesis for the proof.

This can be expressed as a rule of inference as:

$$[P(1) \wedge \forall k P(k) \Rightarrow P(k+1)] \Rightarrow \forall n P(n)$$

# *An example of natural induction*

You are required to prove that  $1+2+3+\dots+n = n(n+1)/2$

We say  $P(k)$  is true iff  $1+2+3+\dots+n = k(k+1)/2$

BASIC STEP:  $P(1)$  is true since  $1 = 1(1+1)/2$

INDUCTIVE STEP: If  $P(k)$  is true then

$$(1)+(2)+(3)+\dots+(k)+(k+1) = k(k+1)/2 + (k+1)$$

$$= (k^2+k)/2 + (k+1)$$

$$= (k^2 + k + 2k + 2)/2$$

$$= (k+1)(k+2)/2$$

So we have established that if  $P(k)$  is true then

$$(1)+(2)+(3)+\dots+(k)+(k+1) = (k+1)(k+2)/2 \text{ i.e. } P(k+1) \text{ is true.}$$

From the base case  $P(1)$  is true. From the inductive step  $P(k) \Rightarrow P(k+1)$  i.e.  $P(1) \Rightarrow P(2)$  and then  $P(2) \Rightarrow P(3)$  and so on. So  $P(k)$  is true for all positive integers  $k$ . Therefore we have proved the relationship using induction.



# *Proof by Strong Natural Induction*

Will be covered in Unit 5