Discrete Mathematics

Unit 2: Arguments, Inferences and Proofs

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Discrete Mathematics Lecture Notes

Acknowledgements

- These lecture notes contain some material from the following sources:
  - *Introduction to Computer Mathematics* by C. Runciman, 2003
  - *Discrete Mathematics and Its Applications* by K. Rosen, 5th Edition, Tata McGraw Hill Edition.

Arguments

An **argument** is an assertion of the form $P_1, P_2 \ldots P_n \vdash Q$, where propositions $P_1, P_2 \ldots, P_n$ are its premises and proposition Q is its conclusion. This can also be written as

$$\begin{array}{c} P_1 \\ P_2 \\ \vdots \\ \underline{P_n} \\ Q \end{array}$$

An argument is **valid** if whenever its premises are true, its conclusion is true; otherwise the argument is **fallacious**.

Valid Arguments?

Consider the following argument:

Socrates is a man

All men are mortal

Socrates is mortal

Is this a valid argument?

Valid Arguments?

Now, consider this argument:

Every foo is a bar
Every bar is a baz
Every foo is a baz

So, is this a valid argument?

Valid Arguments?

Now, consider this argument:

At least one foo is a bar

<u>At least one bar is a baz</u>

At least one foo is a baz

So, is this a valid argument?

Valid Arguments?

Would it be easier if you replaced foo, bar and baz with words that made sense e.g.

At least one student is a Teaching Assistant

At least one Teaching Assistant is a Lab Assistant

At least one student is a Lab Assistant

Does the argument seem valid now?

A Countermodel?

The argument can also be proved not valid using a countermodel without any other knowledge of the domain.

Imagine a world that contains exactly two people:

1. a student who is a TA but not a Lab Assistant
2. a TA who is a Lab Assistant but not a student

Is the previous argument correct in this world?

Valid Arguments

An argument $P_1, P_2 \ldots P_n \vdash Q$ is **valid** if and only if $P_1 \wedge P_2 \wedge \ldots \wedge P_n \Rightarrow Q$ is a tautology (i.e. always true). Otherwise they are called **fallacies**.

Argument 1:

If you read the book, you will pass the course

You read the book

You passed the course

Is this argument valid?

p *You read the book*

q *You passed the course*

Valid or Fallacious?

p *You read the book*

q *You passed the course*

Argument 2:

If you read the book, you will pass the course

You did not read the book

You did not pass the course

Argument 3:

If you read the book, you will pass the course

You passed the course

You read the book

Valid Arguments

Consider the following arguments:

Modus ponens *is the argument* $p, p \Rightarrow q \vdash q$.
Modus tollens *is the argument* $p \Rightarrow q, \neg q \vdash \neg p$.
reductio ad absurdum *is the argument* $p \Rightarrow q, p \Rightarrow \neg q \vdash \neg p$

Are they valid?

Rules of Inference

Rules of inference can also be used to show how the conclusion of an argument can be derived from.

Using such rules one can show how the conclusion must be true if the premises are true, thereby proving that $P_1 \wedge P_2 \wedge \ldots \wedge P_n \Rightarrow Q$ is a tautology and the argument is valid

Rules of Inference

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TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
p $p \to q$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

An example

If you email me, I will finish programming

If you do not email me, I will sleep early

If I sleep early, I will wake up refreshed

If I do not finish programming, I will wake up refreshed

p ≡ You email me

q ≡ I finish programming

r ≡ I sleep early

s ≡ I wake up refreshed

An example

If you email me, I will finish programming

If you do not email me, I will sleep early

If I sleep early, I will wake up refreshed

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Inference Rules with Quantified Statments

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TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\forall x P(x)$ / $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c / $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ / $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c / $\therefore \exists x P(x)$	Existential generalization

An example

A student in this class has not read the book

Everyone in this class passed the first exam

Someone who passed the first exam hasn't read the book

$C(x)$ x is in this class

$B(x)$ x has read the book

$P(x)$ x has passed the first exam

Are the following arguments valid? Justify your answers using propositional logic.

- If I could find my house keys, I could get into the house. I can't find my keys. *Therefore* I can't get into the house.
- All horses are green. Billy is blue. *Therefore* Billy is not a horse.
- Study is necessary to understand logic. If I stay asleep, I don't study. But I shall not stay asleep. *Therefore* I shall understand logic.
- All academics are introverts. Unless I am an academic, I do not wear a tweed jacket. I am not an introvert. *Therefore* I do not wear a tweed jacket.

Methods of Proof

- Direct Proofs
- Indirect Proofs
- Vacuous Proofs
- Proof by Contradiction
- Proof by Cases
- Existence Proofs
- Counterexamples
- Uniqueness Proofs

Theorems and Proofs

- A **theorem** is a statement that can be shown to be true
- One can demonstrate that a theorem is true with a sequence of statements that for an argument called a **proof**.
- The statements used in a proof include **axioms** or **postulates** (underlying assumptions about mathematical structures), while **rules of inference** tie together the steps of a proof.
- A **fallacy** is argument that is not valid because of incorrect reasoning.
- A **lemma** is a simple theorem used in proving other theorems.
- A **corollary** is a proposition that can be established directly from a theorem that has been proved.
- A **conjecture** is a statement whose truth value is unknown. Once a conjecture is shown to be true via a proof it becomes a theorem
- For an implication $p \Rightarrow q$, p is called the **hypothesis** and q is its **conclusion**.

Direct Proofs

An implication $p \Rightarrow q$, can be proved by showing whenever p is true then q must be true.

For example, consider the following definitions:

A integer n is even if there exists another integer k such that $n=2k$ and it is odd if there exists another integer k such that $n=2k+1$. A number can be either even or odd and not both.

Now we are told to attempt a direct proof of the theorem "If n is an odd integer then n^2 is an odd integer."

Indirect Proofs

Since $p \Rightarrow q$ is equivalent to its contrapositive $\neg q \Rightarrow \neg p$, one can prove it is true by proving that contrapositive is true.

Prove that “If $3n+2$ is odd, then n is odd” via indirect proof

Vacuous and Trivial Proofs

If for an implication $p \Rightarrow q$, its hypothesis p is always false then the statement $p \Rightarrow q$ is always true.

E.g. $P(x) \equiv x^2 > x$ where x is a natural number.

Prove that "$P(0) \Rightarrow \forall n \, P(n)$"

Prove by Contradiction

While proving p is true, suppose that a contradiction q can be found so that $\neg p \Rightarrow q$ is true, that is $\neg p \Rightarrow F$ is true. Then the proposition $\neg p$ must be true. Consequently p is true. One can find a contradiction such that $\neg p \Rightarrow r \wedge \neg r$, therefore p is true.

Show that at least four of any 22 days must fall on the same day of the week.

Prove by Contradiction

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Show that at least four of any 22 days must fall on the same day of the week.

Proof: Let p be the proposition "at least four of any 22 days must fall on the same day of the week". Suppose $\neg p$ is true. i.e. at most three of any 22 days must fall on the same day of the week. Since a week has 7 days, this implies at most 21 days are picked. Therefore, we have a contradiction, making $\neg p \Rightarrow r \wedge \neg r$ true, where r is the statement "22 days were chosen". Therefore p is true.

Proof by Cases

To prove an implication of the form:

$(p_1 \vee p_2 \vee \ldots p_n \Rightarrow q)$

Since we know

$[(p_1 \vee p_2 \vee \ldots p_n \Rightarrow q)] \Leftrightarrow [(p_1 \Rightarrow q) \wedge (p_2 \Rightarrow q) \wedge \ldots (p_n \Rightarrow q)]$

All we have to do is prove all $p_i \Rightarrow q$ are true where $i = 1, 2, \ldots, n$

For example: Use a proof by cases to show that $|xy| = |x||y|$ where x and y are real numbers. Remember that $|x|=x$, if $x \geq 0$ and $|x|=-x$ if $x<0$

Prove of Equivalence

In order to prove that $p \Leftrightarrow q$, one can use the tautology:

$$p \Leftrightarrow q \equiv p \Rightarrow q \wedge q \Rightarrow p$$

So by individually proving $p \Rightarrow q$ is true and $q \Rightarrow p$ is true, one can prove $p \Leftrightarrow q$ is true as a whole.

Witnesses and Counter-examples

Existential Proof

For an existential formula, $\exists x, p(x)$ a **witness** is a value of x making $p(x)$ true, thereby proving $\exists x, p(x)$ true as a whole.

Proof by Counter-example

For a universal formula, $\forall x, p(x)$ a **counter-example** is a value of x making $p(x)$ false, thereby proving $\forall x, p(x)$ false as a whole.

Uniqueness Proof

Existence: We show that an element x with the desired property exists, i.e. $\exists x\, p(x)$

Uniqueness: We show that if $y \neq x$, then y can not have the desired property, i.e. $\exists x\, p(x) \wedge \forall y\, (x \neq y) \Rightarrow \neg p(x)$
or $\exists x\, p(x) \wedge \forall y\, p(y) \Rightarrow y = \mathrm{x}$

Example: Show that every integer has an additive inverse, i.e. show that for every integer p there is a unique integer q such that $p+q=0$

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Example: Show that every integer has an additive inverse, i.e. show that for every integer p there is a unique integer q such that $p+q=0$

Proof: If p is an integer, we find that p+q=0 when q=-p and q is also an integer. Thus, we have proven the existence part.

To show that q is unique, suppose that r is an integer with $r \neq q$ such that $p+r=0$. Then $p+r=p+q$. By subtracting p from both sides we have $q=r$, which contradicts our assumption $r \neq q$. Consequently, there is a unique integer q such that $p+q=0$

Proof by Weak Natural Induction

A proof by mathematical induction that $P(n)$ is true for every positive integer n consists of two steps:

BASIC STEP: The proposition $P(1)$ is shown to be true.

INDUCTIVE STEP: The implication $P(k) \Rightarrow P(k+1)$ is shown to be true for every positive integer k.

$P(k)$ is called the inductive hypothesis for the proof.

This can be expressed as a rule of inference as:

$$[P(1) \wedge \forall k\, P(k) \Rightarrow P(k+1)] \Rightarrow \forall n\, P(n)$$

An example of natural induction

You are required to prove that $1+2+3+\ldots+n = n(n+1)/2$
We say $P(k)$ is true iff $1+2+3+\ldots+n = k(k+1)/2$

BASIC STEP: $P(1)$ is true since $1 = 1(1+1)/2$
INDUCTIVE STEP: If $P(k)$ is true then
$(1)+(2)+(3)+\ldots+(k)+(k+1) = k(k+1)/2 + (k+1)$
$= (k^2+k)/2 + (k+1)$
$= (k^2 + k + 2k + 2)/2$
$= (k+1)(k+2)/2$
So we have established that if $P(k)$ is true then
$(1)+(2)+(3)+\ldots+(k)+(k+1) = (k+1)(k+2)/2$ i.e. $P(k+1)$ is true.

From the base case $P(1)$ is true. From the inductive step $P(k) \Rightarrow P(k+1)$ i.e. $P(1) \Rightarrow P(2)$ and then $P(2) \Rightarrow P(3)$ and so on. So $P(k)$ is true for all positive integers k. Therefore we have proved the relationship using induction.

Proof by Strong Natural Induction

Will be covered in Unit 5