Discrete Mathematics

Unit 1: Propositional and First Order Logic

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Discrete Mathematics Lecture Notes

Acknowledgements

- These lecture notes contain some material from the following sources:

  [ICM] C. Runciman: *Introduction to Computer Mathematics,* University of York, 2003

  [Rosen] K. Rosen: *Discrete Mathematics and Its Applications,*, 5th Edition, Tata McGraw-Hill, 2002

Some definitions

A **statement** is a collection of symbols that has a truth value – either false (F) or true (T).

E.g. London is in UK, 1+1=5

Variables in propositional logic are symbols that have a certain meaning and take a truth value depending on their interpretation and the model. E.g. *p* (where *p* stands for ‘John was present in class today’)

Connectives are symbols that are used to form larger statements out of smaller ones.

Some Basic Connectives

A **disjunction** is a compound statement in which two substatements are connected by $\vee$ ('or').

A **conjunction** is a compound statement in which two substatements are connected by $\wedge$ ('and').

The **negation** of statement p is $\neg p$, meaning 'p is not the case'.

The '**implies**' connective $p \Rightarrow q$ can be read as 'if p then q' or 'p guarantees q' or 'p is sufficient for q'.

The '**if and only if**' connective $p \Leftrightarrow q$ reads 'p is both necessary and sufficient for q'.

Truth Tables

A truth table is a mathematical table used in logic to compute the functional values of logical expressions on any of their functional arguments, that is, with respect to the various possible combinations of values that their logical variables may take.

Remember that with *n* logical variables, the truth table will always have 2^n rows.

Truth Tables: Examples

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

p	$\neg p$
F	T
T	F

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Truth Tables: Examples

p	q	$p \Rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

p	q	$p \Leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Propositions

A **proposition** is a statement in which basic substatements are variables each with *F* and *T* as possible values.

Example: p, $\neg p$, $p \wedge q$, $p \vee \neg q$, etc.

Two propositions x and y are **logically equivalent** ($x \equiv y$) if the final columns of their truth tables are identical, i.e. x and y have the same truth value regardless of the values of their variables.

Example: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Tautologies and Contradictions

Tautology: A proposition that is always true regardless the value of its variables. The truth table of a tautology will contain only T's in the final column, i.e. if x is a tautology then $x \equiv T$

E.g. $p \vee \neg p$, $p \vee T$

Contradiction: A proposition that is always false regardless of the value of its variables. The truth table of a contradiction will contain only F's in the final column, i.e. if x is a contradiction then $x \equiv F$

E.g. $p \wedge \neg p$, $p \wedge F$

Practice with Logic

Put the following sentences into propositional logic:

1. Ed is wise
2. Ed is not wise
3. Ed is not wise and Ed is not well
4. It's not true that Ed is wise and well
5. If Ed dies he will miss class
6. Either Ed flies and he arrives on time or Ed walks and he is not on time

Symbols to use:

W Ed is wise
C Ed attends class
L Ed is well
D Ed is Dead
F Ed flies
A Ed arrives on time
K Ed Walks
$\neg, \vee, \wedge, \rightarrow,$ and $\leftrightarrow$

Connectives and Rules in Propositional Logic

high priority	
$\neg$	not
$\wedge$	and
$\vee$	or
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if
low priority	

xor	**exclusive or**	$p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$
$\downarrow$	**negated or**	$p \downarrow q \equiv \neg(p \vee q)$
$\uparrow$	**negated and**	$p \uparrow q \equiv \neg(p \wedge q)$
$\Rightarrow$	**implication**	$p \Rightarrow q \equiv \neg p \vee q$
$\Leftrightarrow$	if and only if	$p \Leftrightarrow q \equiv (q \Rightarrow p) \wedge (p \Rightarrow q)$

$$\left.\begin{aligned} p \vee p &\equiv p \\ p \wedge p &\equiv p \end{aligned}\right\} \textit{idempotence}$$

$$\left.\begin{aligned} (p \vee q) \vee r &\equiv p \vee (q \vee r) \\ (p \wedge q) \wedge r &\equiv p \wedge (q \wedge r) \end{aligned}\right\} \textit{associativity}$$

$$\left.\begin{aligned} p \vee \neg p &\equiv T \\ p \wedge \neg p &\equiv F \end{aligned}\right\} \textit{excluded middle}$$

$$\left.\begin{aligned} p \vee q &\equiv q \vee p \\ p \wedge q &\equiv q \wedge p \end{aligned}\right\} \textit{commutativity}$$

$$\left.\begin{aligned} p \vee F &\equiv p \\ p \wedge T &\equiv p \end{aligned}\right\} \textit{identity}$$

$$\left.\begin{aligned} p \vee (q \wedge r) &\equiv (p \vee q) \wedge (p \vee r) \\ p \wedge (q \vee r) &\equiv (p \wedge q) \vee (p \wedge r) \end{aligned}\right\} \textit{distributivity}$$

$$\left.\begin{aligned} p \vee T &\equiv T \\ p \wedge F &\equiv F \end{aligned}\right\} \textit{strictness}$$

$$\left.\begin{aligned} \neg(p \vee q) &\equiv (\neg p) \wedge (\neg q) \\ \neg(p \wedge q) &\equiv (\neg p) \vee (\neg q) \end{aligned}\right\} \textit{de Morgan}$$

$$\neg\neg p \equiv p \quad \textit{double negation}$$

$$\left.\begin{aligned} p \vee (p \wedge q) &\equiv p \\ p \wedge (p \vee q) &\equiv p \end{aligned}\right\} \textit{absorption}$$

From [ICM]

An example proof of equivalence

$$p \wedge (p \Rightarrow q) \Rightarrow q \equiv T$$

$$
\begin{aligned}
& p \wedge (p \Rightarrow q) \Rightarrow q \\
\equiv \; & \{ \text{ translation of } \Rightarrow \} \\
& \neg(p \wedge (\neg p \vee q)) \vee q \\
\equiv \; & \{ \text{ distribute } \wedge \text{ over } \vee \} \\
& \neg(p \wedge \neg p \vee p \wedge q) \vee q \\
\equiv \; & \{ \text{ excluded middle; identity } \} \\
& \neg(p \wedge q) \vee q \\
\equiv \; & \{ \text{ de Morgan } \} \\
& \neg p \vee \neg q \vee q \\
\equiv \; & \{ \text{ excluded middle; strictness } \} \\
& T
\end{aligned}
$$

From [ICM]

Proving other propositions

Definition *For any implication $p \Rightarrow q$:*

its **converse**	*is*	$q \Rightarrow p$;
its **inverse**	*is*	$\neg p \Rightarrow \neg q$;
its **contrapositive**	*is*	$\neg q \Rightarrow \neg p$;

original	$p \Rightarrow q$	$\equiv$	$\neg q \Rightarrow \neg p$	contrapositive
converse	$q \Rightarrow p$	$\equiv$	$\neg p \Rightarrow \neg q$	inverse

Prove the given propositions using previously described theorems

From [ICM]

Boolean Searches

Logical connectives are used extensively in searches of large collections of information, e.g. indexes of web pages.

E.g. If you do a web search for the following:

Search for “Syedur” AND “Rahman”

Search for “Syedur” OR “Rahman”

Search for “Syedur” AND “RAHMAN” AND NOT “MOHAMMED”

Logic Puzzle

"Puzzle that can be solved using logical reasoning"

Smullyan's puzzle:

There is an island with two kinds inhabitants: knights (who always tell the truth) and knaves (who always lie).

You come across two inhabitants *A* and *B*. *A* claims that "*B* is a knight" while *B* claims "the two of us are of opposite kinds"

What kind are *A* and *B?*

Logic and Bit Operations

A bit (binary digit) has two values 0 or 1 representing a Boolean variables of values F and T respectively.

A bit string is a sequence of zero or more bits. E.g. 0, 01

Bit operations correspond to logic operations.

Example bit-wise AND, OR and NOT operations:

	000111		011000		
AND	011010	OR	010111	NOT	10010
	000010		011111		01101

First Order Predicate Logic

A **predicate** is a proposition $p(v_1, v_2, \ldots v_n)$ depending on variables $v_1, v_2, \ldots v_n$. Given a value for each v_i, p defines a statement that is true or false.

Examples:

- *even*(*x*) ≡ '*x* is an even number'
- *divides*(*x*,*y*) or its short form $x|y$ ≡ '*x* divides *y*'
- *age(s,x)* ≡ '*s* is *x* years old'
- *adult*(*s*) ≡ '*s* is at least 18 years old'

Quantifiers in First Order Logic

$\forall$ **Universal quantifier**: A formula $\forall x, p(x)$ reads 'for all values of x in a particular universe of discourse or domain, $p(x)$ is true'.

$\forall x$,dishonest(x) reads 'everyone/thing is dishonest' but if the domain of x is specified as politicians then it reads "all politicians are dishonest'

$\forall x,\ horse(x) \Rightarrow quadruped(x)$ reads 'if x is a horse then it is a quadruped' or in other words 'all horses are quadrupeds'

$\exists$ **Existential quantifier**: A formula $\exists x, p(x)$ reads 'there exists a value of x such that $p(x)$ is true'.

$\exists$x, *horse*(x) reads 'there is a horse'

$\exists$x, *horse*(x) $\wedge$ *colour*(x, *black*) reads 'there is a horse which is black'

Binding Variables: A variable x is bound, when a quantifier is used on the variable, otherwise it is free.

In $\forall x, p(x,y)$, x is bound but y is free.

Another Example

Remember the example:

age(s,x) ≡ ‘*s* is *x* years old’

adult(*s*) ≡ ‘*s* is at least 18 years old’

How do we define *adult*(*s*) using logic?

Another Example

Remember the example:

age(s,x) ≡ '*s* is *x* years old'

adult(*s*) ≡ '*s* is at least 18 years old'

How do we define *adult*(*s*) using logic?

adult(*s*) ≡ *s* has an age $x \wedge x \geq 18$

adult(*s*) ≡ $\exists x, (age(s,x) \wedge x \geq 18)$

Quantifiers and De Morgan

De Morgan's law extends over quantifiers:

$$\exists x, \neg p(x) \equiv \neg(\forall x, p(x))$$

$$\forall x, \neg p(x) \equiv \neg(\exists x, p(x))$$

Example:

$\neg(\exists x, unicorn(x)) \equiv \forall x, \neg unicorn(x) \equiv$ there is no unicorn

Restricting Predicates

Often the range of values for a predicate are restricted:

$\forall x,(p(x) \Rightarrow q(x))$ reads ‘for all x of type p, $q(x)$ is true’.

$\exists x,(p(x) \wedge q(x))$ reads ‘for some x of type p, $q(x)$ is true’.

Extended De Morgan works for restricted predicates too:

$$\neg(\forall x,(p(x) \Rightarrow q(x))) \equiv \exists x, p(x) \wedge \neg q(x)$$

$$\neg(\exists x,p(x) \wedge q(x)) \equiv \forall x, p(x) \Rightarrow \neg q(x))$$

Commutability

Quantifiers may commutate only in case of similar ones.

Therefore the following are true:

$$\exists x, \exists y, p(x,y) \equiv \exists y, \exists x, p(x,y)$$

$$\forall x, \forall y, p(x,y) \equiv \forall y, \forall x, p(x,y)$$

And the following is NOT true:

$$\forall x, \exists y, p(x,y) \equiv \exists y, \forall x, p(x,y)$$

Logic Programming

Declarative languages such as Prolog are used for logic programming, where rather than carrying out a set of instructions (as in procedural languages such as C, Java etc.), programs make inferences given rules and facts.

Note that Prolog variables are in uppercase and constants/predicates are in lowercase.

Knowledge base:	**Prolog Facts:**
Mark is Tom's parent	parent(mark, tom)
Jill is Tom's Parent	parent(jill, tom)
x is y's parent iff y is x's child	child(x,y) :- parent(y,x)
i.e. $\forall x \forall y$ parent(x,y)$\Leftrightarrow$child(y,x)	parent(x,y) :- child(y,x)

Query:	**Prolog Queries:**
Is Tom Jill's child?	child(tom, jill)
Is Jill Tom's child?	child(jill, tom)
Does Jill have children, who are they?	child(X, jill)
Does Tom have children, who are they?	child(X, tom)
Does Tom have parents, who are they?	parents(X,tom)

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Query:	**Prolog Queries:**	**Answers:**
Is Tom Jill's child?	child(tom, jill)	Yes
Is Jill Tom's child?	child(jill, tom)	No
Does Jill have children, who are they?	child(X, jill)	Yes X=tom, No
Does Tom have children, who are they?	child(X, tom)	No
Does Tom have parents, who are they?	parents(X,tom)	Yes X=mark, X=jill, No

Witness and Counter-examples

For an existential formula, $\exists x, p(x)$ a witness is a value of x making $p(x)$ true, thereby proving $\exists x, p(x)$ true as a whole.

For a universal formula, $\forall x, p(x)$ a counter-example is a value of x making $p(x)$ false, thereby proving $\forall x, p(x)$ false as a whole.

Uniqueness

The Uniqueness Quantifier $\exists!$: A formula $\exists! x, p(x)$ reads 'there exists exactly one value of x such that $p(x)$ is true' given a certain domain of x.

$\exists! x, p(x)$ can be expressed as:

$(\exists x\ p(x)) \wedge (\forall y\ \forall z\ (p(y) \wedge p(z) \Rightarrow y=z))$

Or more simply as:

$\exists x\ p(x) \wedge \forall y\ p(y) \Rightarrow y=\mathrm{x}$

and more formally as:

$\exists x\ p(x) \wedge \forall y\ (x \neq y) \Rightarrow \neg p(x)$

Note that sometimes $\exists! x, p(x)$ is written as $\exists_1 x, p(x)$

More examples with quantifiers

Given the domain of real numbers

What do the following mean:

- $\forall x\, \exists y\, (x<y)$
- $\forall x\, \forall y\, \exists z\, (z = x + y)$
- $\forall x\, \forall y\, (x>0 \wedge y<0 \Rightarrow x-y>0)$

Write the following using quantifiers

- The product of two positive numbers is positive.
- Every pair of numbers has a product which is a number
- There is no largest number

More examples with quantifiers

Given *loves*(*x, y*) reads "*x* loves *y*" with the domain of people

What do the following mean?

- $\forall x\, \forall y\, \forall z\;\; loves(x, y) \wedge loves(x, z) \Rightarrow z{=}y$
- $\forall x\, \exists y\; loves(x, y) \wedge \forall z\; (z \neq y) \Rightarrow \neg loves(x, z)$

Using *loves*(*x*, *y*), define a predicate *unloved*(*x*) which reads "*x* is an unloved person", i.e. there is no one that loves *x.*

Expanding Quantifiers

Suppose there are just two makes m of car (Toyota and Ford) and three colours c of car paint (black, silver and red). Expand out the quantifiers in the following formulae to obtain equivalents using $\wedge$ and $\vee$:

(a) $\forall m, \exists c, \text{available}(m, c)$

(b) $\exists c, \forall m, \text{available}(m, c)$

Is (a) $\Rightarrow$ (b) true? Is (b) $\Rightarrow$ (a) true?

From [ICM]

Expand the quantifiers ....

$$\forall m, \exists c, \text{available}(m, c)$$
$$\equiv \quad (\text{available}(Toyota, black) \vee \text{available}(Toyota, silver) \vee \text{available}(Toyota, red)) \wedge$$
$$(\text{available}(Ford, black) \vee \text{available}(Ford, silver) \vee \text{available}(Ford, red))$$

$$\exists c, \forall m, \text{available}(m, c)$$
$$\equiv \quad (\text{available}(Toyota, black) \wedge \text{available}(Ford, black)) \vee$$
$$(\text{available}(Toyota, silver) \wedge \text{available}(Ford, silver)) \vee$$
$$(\text{available}(Toyota, red) \wedge \text{available}(Ford, red))$$

(b) $\Rightarrow$ (a) is true but not the converse.

From [ICM]

A Sample Problem

Let $S(x)$ be the predicate "x is a student", $F(x)$ be "x is a faculty member" and $A(x,y)$ be "x has asked y a question" where the universe of discourse consists of all people associated with your school. Use quantifiers to express each of these statements.

a) Lois has asked Professor Michaels a question.

b) Every student has asked Professor Gross a question.

c) Every faculty member has either asked Professor Miller a question or been asked a question by Professor Miller.

d) Some student has not asked any faculty member a question.

e) There is a faculty member who has never been asked a question by a student.

f) Some student has asked every faculty member a question.

g) There is a faculty member who has asked every other faculty member a question.

h) Some student has never been asked a question by a faculty member.

From [Rosen]